Also solved by O. S. ADAMS, C. N. SCHMALL, PAUL CAPRON, and HORACE OLSON.

MECHANICS.

324. Proposed by H. S. UHLER, Yale University.

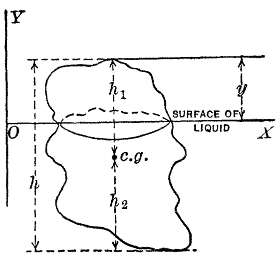
A rigid body of any shape is at rest in a neutral liquid which is also at rest and has an indefinitely great volume. The body is so situated that the free surface of the liquid is tangent to it at its highest point (or points). All the space above the liquid is filled with a neutral stagnant fluid whose density is not greater than the density of the liquid. Show that the work done in raising (pure translation) the body very slowly until the interface of the two fluids is tangent to it at its lowest point (or points) is expressible by the formula $mgh - gV(\rho_1h_1 + \rho_2h_2)$, where $m \equiv$ mass of body, $V \equiv$ volume of body, $\rho_1 \equiv$ mean density of lower medium, $\rho_2 \equiv$ density of upper medium, $h_1 \equiv$ distance of center of mass of the displaced liquid below the free surface in the initial position of the body, $h_2 \equiv$ elevation of center of mass of displaced fluid above the interface in final position of the body and $h = h_1 + h_2$. (Neglect surface-tension, etc.)

SOLUTION BY J. B. EPPES, Annapolis, Md.

Take the axes indicated in the figure. Let the highest point of the body at any time be a distance y above the surface of the liquid, and let A be area of the cross-section of the body made by the surface of the liquid.

Then the mass of the fluid displaced is $\rho_2\int_0^y Ady$ and the mass of the liquid displaced is $\rho_1\left(V - \int_0^y Ady\right)$.

Hence, the downward force is $g\left[m - \rho_2\int_0^y Ady - \rho_1\left(V - \int_0^y Ady\right)\right]$. Then, the total work in raising the body, is



$$\int_0^h\left[mg - \rho_2 g\int_0^y Ady - \rho_1 g\left(V - \int_0^y Ady\right)\right]dy = mgh$$
$$- \rho_2 g\int_0^h dy\int_0^y Ady - \rho_1 gVh + \rho_1 g\int_0^h dy\int_0^y Ady.$$

Now assume A as a function of y to be of the form

$$A \equiv B + Cy + Dy^2 + \cdots.$$

Then,

$$\int_0^h dy\int_0^y Ady = \int_0^h\left(By + \frac{Cy^2}{2} + \frac{Dy^3}{3} + \cdots\right)dy = \frac{Bh^2}{2} + \frac{Ch^3}{2\cdot 3} + \frac{Dh^4}{3\cdot 4} + \cdots$$
$$= \left[Bh^2 + \frac{Ch^3}{2} + \frac{Dh^4}{3} + \cdots\right] - \left[\frac{Bh^2}{2} + \frac{Ch^3}{3} + \frac{Dh^4}{4} + \cdots\right]$$
$$= h\int_0^h Ady - \int_0^h yAdy = hV - Vh_1 = Vh_2.$$

Hence, work $= mgh - \rho_2 gVh_2 - \rho_1 gVh + \rho_1 gVh_2 = mgh - gV(\rho_1h_1 + \rho_2h_2)$.

325. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The lever of a testing machine is c feet long, and is poised on a knife edge a feet from one end and b feet from the other, and in a horizontal line above which the beam is symmetrical. The beam is m inches deep at the knife edge and tapers uniformly to a depth of n inches at each end; the width of the beam is the same throughout its length. Find the distance of the center of gravity of the beam from the knife edge.

SOLUTION BY W. J. THOME, University of Detroit.

It is a well known fact that if the parallel sides of a trapezoid are g and s, and the perpendicular distance between them is p, then the distance from the greater side g to the center of gravity of the trapezoid is

$$\frac{p}{3}\frac{(g+2s)}{(g+s)}.$$

Let t be the constant width of the lever, in inches;
d, the density of the material of the lever,
M, the total mass of the lever;
m_1, the mass of that part of the lever from the knife edge to the end distant a feet away;
m_2, the mass of that part of the lever from the knife edge to the end distant b feet away;
$\overline{X}$, the distance in feet from the knife edge to the center of gravity of the entire lever;
$\bar{x}_1$, the distance in feet from the knife edge to the center of gravity of the mass m_1; and
x_2, the distance in feet from the knife edge to the center of gravity of the mass m_2.
Then, using the foot as our unit of length, we have

$$M\overline{X} = m_2\bar{x} - m_1\bar{x}_1;$$

or

$$\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d + \frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\overline{X}$$

$$= \left[\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\left[\frac{b}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right] - \left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d\right]\left[\frac{a}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right].$$

Hence,

$$\overline{X} = \frac{1}{3}\frac{(m+2n)}{(m+n)}(b-a),$$

which is to be measured in the direction of whichever is the greater, b or a.

NUMBER THEORY.

240. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^3 - px + q = 0$ are rational, prove that $4p - 3x^2$ is a perfect square.

SOLUTION BY WILLIAM E. PATTEN, Government Institute of Technology, Shanghai, China.

Let the roots of $x^3 - px + q = 0$ be x_1, x_2, x_3. Then

$$x_1 + x_2 + x_3 = 0;$$

and

$$x_1x_2 + x_2x_3 + x_3x_1 = -p.$$

Therefore,

$$\begin{aligned} 4p - 3x_1^2 &= -4(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1^2 \\ &= -4x_2x_3 - 4x_1(x_2 + x_3) - 3x_1^2 \\ &= -4x_2x_3 - 4(-x_2 - x_3)(x_2 + x_3) - 3(-x_2 - x_3)^2 \\ &= -4x_2x_3 + (x_2 + x_3)^2 = (x_2 - x_3)^2. \end{aligned}$$

Since the coefficient of the leading term of the given equation is unity, and the roots are rational, they are also integral.

Therefore, $x_2 - x_3$ is an integer, and $4p - 3x_1^2$ is a perfect square.

Similarly for the other roots:

$$4p - 3x_2^2 = (x_3 - x_1)^2, \qquad \text{and} \qquad 4p - 3x_3^2 = (x_1 - x_2)^2.$$